

# The Axiomatic melting pot

*Teaching probability theory in Prague during the 1930's*

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## Résumé

Dans cet article, nous nous intéressons à la façon dont la théorie des probabilités a été enseignée à Prague et en Tchécoslovaquie, notamment durant les années 1930. Nous portons une attention toute particulière à un livre de cours de probabilités, publié à Prague par Karel Rychlík en 1938, et qui utilise comme support l'axiomatisation de Kolmogorov, un fait très exceptionnel avant la deuxième guerre mondiale.

## Abstract

In this paper, we are interested in the teaching of probability theory in Prague and Czechoslovakia, in particular during the 1930's. We focus specially on a textbook, published in Prague by Karel Rychlík in 1938, which uses Kolmogorov's axiomatization, a very exceptional fact before World War II.

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## INTRODUCTION

It is now obvious that the first half of the 20th century was a golden age for probability theory. Between the beginning of Borel's (1871–1956) interest in the field in 1905 and the rapid development of the general theory of stochastic processes and stochastic calculus in the 1950's, mathematical probability evolved from a rather small topic, with interesting but quite scattered results, to a powerful theory with precise foundations and an active field of investigation. There is a huge literature dealing with the description of this burst of interest. Let us just mention Von Plato's book [74] whose subject is precisely to describe this new emergence of probability and renewal of the theory after the creations of the 17-18th centuries and the Laplacian impulse of the beginning of the 19th. The book contains a huge bibliography in which the reader may find a lot of interesting information to complete the picture. The years which follow immediately the First World War are particularly crucial, as they appear to have been the moment when the consciousness of the importance and power of set measure theory, intuited by Borel twenty years ago, appeared more clearly to a series of mathematicians, in particular those of the next generation. Among them, the young Kolmogorov (1903–1987) made several decisive steps and published the celebrated treatise [40] where he fixed an axiomatization of probability in the framework of measure theory, an axiomatization which is still used today as the basis of the mathematical theory of probability. It is too often considered though that this axiomatization was accepted immediately and without discussions. On the contrary, the period between 1920 and 1940 was a moment of

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intense debates about the meaning of probability and the possible solutions for giving a reasonable axiomatization to it. The passionate discussions in the Berlin and Vienna Circles around Reichenbach and Carnap belong precisely to this moment. The reader may consult the excellent paper [70] to have a better idea of these debates and a survey of the state of probability theory during these years.

The years following the First World War were also the moment when new born countries, having acquired independence after the implosion of the Central Empires and Czarist Russia, were looking for new strategies to improve their scientific contacts and their presence in the scientific movement. The example of Czechoslovakia is very interesting in this prospect. Born in 1918, the new state paid a particular attention to its cultural and scientific national and international life, created two universities in Brno and Bratislava, and intended to intensify its contacts with countries such as France and Great-Britain, and at the same time to federate the scientific cooperation with the East-European countries. As far as mathematical probability theory is concerned, the role of Bohuslav Hostinský (1884–1951) was determinant. It was studied in particular in [27] where his mathematical and social activity is considered through the prism of his correspondence with the French mathematician Fréchet (1878–1973).

Therefore, Czechoslovakia during the 1920's and 1930's appeared to be a double barycenter, both in the geographical sense of the world and in the space of mathematical ideas. Prague was the middle point between Berlin and Vienna and therefore a natural point to discuss the diverse interpretations of probability which were set at this moment. An emblematic international conference of philosophy of probability was held in 1929 in Prague where all the tenors were present. But the country was also the middle point between Paris and Moscow at the precise moment when the nervous center of investigation in the field passed from the first to the second. It has already been observed, in particular in [5], that this particular situation at this particular moment played a part in the interest and decisive role of Hostinský in Markov Chains investigation.

In this paper, we are interested in another point of view, the question of teaching probability theory. The general subject of the appearance of probability theory in the Czech lands has already been considered in several studies such as [45] and [35]. In this paper, we focus on the question of teaching probability in the 1930's in Prague. As already said, the question of axiomatization was very urgent during these years, and Prague was a natural place for new educational experiments. We consider this hypothesis through the amazing example of a textbook published by a professor of the Czech Technical University in Prague, Karel Rychlík in 1938, using Kolmogorov axiomatic presentation. To the best of our knowledge Rychlík's publication was the only textbook of probability, with Cramer's one, to use Kolmogorov's axiomatization before 1939. These attempts remained however without posterity and did not survive to the war.

The paper is organized as follows. In a first section, we rapidly expose the state of probability theory teaching in the Czech lands until 1938, by presenting its appearance in the syllabus and the main characters who took care of the lectures, such as the great mathematician Emanuel Czuber. Then, in a second part, we focus on Czech textbooks on probability which appeared in the given period. The third part deals with the biography of Karel Rychlík, the fourth is devoted to a rather detailed description of Rychlík's textbook, and the fifth one to the way it was received and commented on, in particular in several studies by his follower and assistant Otomar Pankraz.

## 1. TEACHING PROBABILITY THEORY IN THE CZECH LANDS

The history of probability in the Czech land has been treated in the book *Vývoj teorie pravděpodobnosti v českých zemích do roku 1938* (*The Development of Probability Theory in the Czech Lands until 1938*) [45] by Karel Mačák so far. This book, published in the series *Dějiny matematiky* (*The History of Mathematics*), is not aimed at describing all stages of the development of this mathematical field in great detail but is devoted to some selected problems. It collects Mačák's lectures on the development of probability theory which he had read in the previous ten years. Quite surprisingly, the book finishes with the very publication of Rychlík's textbook [67], and unlike the present paper does not deal with the following years.

Probability theory, or more precisely probability calculus, until the 1920's in the Czech and Slovak lands, was related especially to teaching mathematics at universities, and apart from some exceptions there was no original scientific work in this field. As the number of mathematics professors at the Prague University was rather small,<sup>4</sup> and their main role was to give basic lectures on algebra, mathematical analysis and geometry, probability theory is to be found in the syllabuses of the lectures only exceptionally.<sup>5</sup> The situation considerably changed by increasing the number of teachers and especially by introducing the studies of actuarial mathematics and mathematical statistics after WW I. Shortly before this, in 1911, Václav Láška (1862–1943) had come to Prague from Lvov and had been appointed as Applied Mathematics professor. In 1912 he had already read a one-hour public lecture *The Introduction to Probability Theory*.<sup>6</sup>

A two-year study programme of actuarial mathematics and mathematical statistics was organized at the Prague University from the academic year 1922/23. The leading personality who took care of teaching in both these subject was Emil Schoenbaum (1882–1967).<sup>7</sup> The lectures on probability theory of this course were read by Miloš Kössler (1884–1961).<sup>8</sup>

The situation in teaching probability at Austrian technical universities was similar. Until the end of the 19th century there were, normally, no special lectures given on this subject and they were

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<sup>4</sup>In 1882 the Prague University was divided into the Czech and German Universities.

<sup>5</sup>The first professor to read lectures on probability from time to time was Wilhelm Matzka (1798–1891) in the 1850's and 1860's.

<sup>6</sup>Láška is an author of more than 300 works on many areas of applied mathematics, including the textbook *Probability Calculus* [42] and the work *Selected Chapters of Mathematical Statistics* [43] which he certainly used in the above mentioned course.

<sup>7</sup>Schoenbaum studied actuarial mathematics at the University of Göttingen in 1906 and following the request of T. G. Masaryk he started working in the field of social insurance. In 1919 he habilitated for actuarial mathematics and mathematical statistics at the Prague University where he was appointed full professor of actuarial mathematics in 1923. In 1930 he became one of the initiators of founding the journal *Aktuárské vědy* (*Actuarial Sciences*). After 1939 he worked abroad where he took part in reforms of social insurance, e.g. in some Latin American countries, but also in the U.S.A. and Canada. He died in Mexico in 1967.

<sup>8</sup>Kössler was appointed extraordinary professor of mathematics in 1922. His scientific work focused mainly on number theory and he did not publish any results of his own on probability theory.

read only as optional lectures by private docents.<sup>9</sup> One of them was Emanuel Czuber (1851–1925)<sup>10</sup> who became a private docent in 1876 when he habilitated in the field of the theory and practice of adjustment. Later he extended his *veniae docendi* to probability theory.

In 1903 the first edition of his book *Die Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung* [15] was published, a huge work which was later divided into two volumes (altogether more than 900 pages), the part on probability was published in the sixth edition as late as 1941. Thus, this book influenced teaching probability not only in German speaking countries for nearly 40 years, it was a basic textbook for Czech mathematicians for a long time as well. We may suppose that it was used mainly for preparation of future workers in actuary.<sup>11</sup>

Czuber was engaged also in historical and philosophical questions of probability theory. In 1899 he published a nearly 300 pages study [13] in *Jahresbericht der Deutschen Mathematiker-Vereinigung* where he described the development of probability theory in the 18th and 19th centuries.<sup>12</sup> Two years before his death the book *Die philosophischen Grundlagen der Wahrscheinlichkeitsrechnung* [17] was published.

Private docent Augustin Pánek (1843–1908) started reading lectures on probability theory at the Czech Technical University of Prague in the 1870's.<sup>13</sup> He read lectures on probability theory until his death, which means even in the period when the actuarial technical course was open. The

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<sup>9</sup>In the middle of the 19th century there existed special study programmes at technical universities for a certain period where "political arithmetic" was taught in which probability and statistics played a significant role.

<sup>10</sup>Emanuel Czuber was born on 19th January, 1851 in Prague where he graduated at the German Technical University in 1874. Czuber, with original surname Čubr, came from a completely Czech family, nevertheless he obtained his education at German schools. When he started teaching in Vienna, he used only the German language for all conversations. In 1872–1875 he was an assistant of geodesy professor Kořistka. Then he became a mathematics and descriptive geometry teacher at a Prague grammar school. In 1876 he habilitated and as a private docent he read two-hour lectures on probability theory, the least square method and mathematical statistics. In 1886–1891 Czuber was mathematics professor at the Technical University in Brno, then at the Technical University in Vienna until his retirement in 1921. He died on 22nd August 1925 in Gnigl at Salzburg. More detailed information on the life and scientific work of Emanuel Czuber, including the list of his publication, can be found e.g. in [19]. Nevertheless a detailed analysis of Czuber's scientific life, his position within Austrian and German mathematical community has not been done yet. During his life Czuber was given a number of significant functions in the area of education as well as actuary. It was him who initiated actuarial technical studies in Austro-Hungary, – details will be given in the following part. In 1898–1900 he was the president of the Association of Austrian-Hungarian actuarial technicians; from 1900 the chair of the Association of Austrian-Hungarian insurance companies; in 1909 he was the chair of the VIth International Congress for Actuarial Sciences in Vienna; in 1890/91 he was the Rector of the Technical University in Brno, in 1894/95 the Rector of the Technical University in Vienna; in 1876–1886 the editor of *Technische Blätter*; in 1897–1921 the editor in chief of *Zeitschrift für Realschulwesen*; the Chair of the Austrian section of the International Committee for teaching mathematics, etc. His scientific work includes especially works on probability theory, the theory of errors and adjustment, geodesy and actuary. Shortly after his habilitation he translated lectures on probability theory of A. Meyer from Lutyk [10] into German and five years later he wrote the first probability textbook of his own *Geometrische Wahrscheinlichkeiten und Mittelwerte* [11] which was published also in French. During his stay in Brno he prepared the book [12].

<sup>11</sup>Czuber published two other books in 1920's, [16, 18].

<sup>12</sup>The list of literature in the study includes around 500 works and it can be considered the second most significant work on the history of probability theory, the first being Todhunter's book [72]. Czuber also contributed to *Encyklopädie der mathematischen Wissenschaften* with the paper [14].

<sup>13</sup>In 1872 Augustin Pánek (1843–1908) habilitated for the field of integrals, nevertheless he gave mainly lectures on probability theory and the least squares method until his appointing an extraordinary professor in 1896. Detailed information about Augustin Pánek can be found in the work of M. Bečvářová [1].

syllabus of Pánek's two-hour lectures can be found in the list of lectures at the Czech Technical University:

Absolute, relative and complex probability. Geometric probability. Bernoulli and Poisson Theorems. Objective and subjective expectations. Probability a posteriori. Bayes Formula. Laplace Theorem. On insurance. Probability and judgment. Historical overview of probability calculus and the least squares method.<sup>14</sup>

Pánek never attempted to write a textbook on probability theory, but his works include articles on probability theory. Those, more or less popularizing, papers published in *Časopis pro pěstování matematiky a fysiky* might indicate what Pánek was teaching. We do not find there, apart from some exceptions, any general considerations, they include in fact solving some simple or more difficult probability exercises.

An important impulse for the development of probability theory teaching came from opening the actuarial technical courses at the Austrian–Hungarian technical universities at the turn of the century. Those courses existed in Austria as well as at Czech and German technical universities in Czechoslovakia until the end of 1930's.

The first such course was established by Emanuel Czuber at the Vienna Technical University in the academic year 1894/95. A very important part of the courses was covered by mathematical subjects. Apart from basic mathematics lectures, common to other study programmes, the students attended lectures on actuarial mathematics, mathematical statistics, and probability theory. The course was open as a three-year programme, but it was shortened to two years already in 1897. It became a model for other schools and despite considerable efforts to provide it as a four-year study programme equal to other courses at technical university, it did not happen so. Probability theory at the Vienna Technical University was read by Emanuel Czuber from 1891 until the end of WW I.<sup>15</sup> His lecture took two hours, after a three-hours opening of the course. It was included into the second year and had the following syllabus:

Notion of probability. Direct determination of probability. Indirect determination of probability. Repeated trials - theorems of Bernoulli and Poisson. Mathematical expectation, mathematical risk. Probability of causes and of future events on the basis of experience. Bases of error theory. Least squares method. Theory of collective measure.<sup>16</sup>

Mathematical statistics which was taught three hours during the whole course was taught by Ernst Blaschke (1856–1926), the author of the textbook [2], in 1896–1926. Actuarial mathematics was read by Viktor Sersavy (1848–1901) until 1901 and from 1901 to 1938 by Alfred Tauber (1866–1942).<sup>17</sup>

<sup>14</sup>Absolutní, relativní a složitá pravděpodobnost. Geometrie pravděpodobnosti. Věta Bernoulliho a Poissonova. Objektivní a subjektivní naděje. Pravděpodobnost a posteriori. Pravidlo Bayesovo. Theorem Laplaceův. O pojišťování. Pravděpodobnost o seznání svědků. Dějinný nástin počtu pravděpodobnosti a metody nejmenších čtverců.

<sup>15</sup>In 1920 a position of probability theory professor was established which was not, however, occupied and until the end of the war it was only substituted.

<sup>16</sup>Wahrscheinlichkeitsbegriff. Direkte Wahrscheinlichkeitsbestimmung. Indirekte Wahrscheinlichkeitsbestimmung. Wiederholte Versuche — die Sätze von Bernoulli und Poisson. Mathematische Hoffnung und mathematische Risiko. Wahrscheinlichkeit von Ursachen und künftigen Ereignissen auf Grund der Erfahrung. Elemente der Fehlertheorie. Methode der kleinsten Quadrate. Kollektivmaßlehre.

<sup>17</sup>More detailed information about actuarial mathematics teaching in Vienna can be found in [63, 48].



The second school where the course was established, in 1904, was the Prague Czech Technical University. It was initiated by Gabriel Blažek (1842–1910) who had been reading lectures on actuary mathematics since 1901. The lectures were then taken over by Josef Beneš (1859–1927) who was appointed a professor in this field after WW I and was also teaching mathematical statistics. After Beneš's death actuary mathematics teaching was taken over by Jaroslav Janko (1893–1965).

Probability was first taught two hours the whole second year, then four hours in the winter semester only. Probability theory was not examined at the final examination, students only had to prove their knowledge during the studies. The two-hour lectures were given by professor Pánek until his death, then by his successor František Velíšek (1877–1914) who read lectures on probability four hours in winter semester. After Velíšek died on the front in 1914, his teaching was taken over by Karel Rychlík whose duties included lecturing on probability theory even after he was appointed professor in 1920. From the academic year 1933/34 Rychlík gave two-hour lectures *Introduction to probability calculus and mathematical statistics* in summer semester, however, this lecture was intended for students of other programmes. Unfortunately, the content of his lectures is not known.

Mathematical statistics was first lectured as an independent subject three hours during the whole second year, then it became part of actuarial mathematics lectures, and afterwards an independent subject again. The syllabus at the beginning of the 1930's was the following:

The history of statistics. The methods of statistical investigation. Describing units from the point of view of qualitative and quantitative indicators. Measures of variance. The theory of index numbers. Mortality tables and their construction. Intensive and time measures, their calculation from a given material. Foundations of adjustment theory. The methods of statistical research on causal relations. The stability of statistic numbers. Stochastic dependence between qualitative and quantitative indicators. The analysis of time series.<sup>18</sup>

Very shortly after having introduced actuarial technical courses at the Czech Technical University similar course was opened also at the German Technical University in Prague. It was in 1906 and actuarial mathematics was taught by Gustav Rosmanith (1865–?) until the mid 1930's. He taught also mathematical statistics which was taken over by his successor Josef Fuhrich (1897–?). Two-hour lectures on probability theory were read by Karl Carda (1870–1943). The syllabus of his lectures at the end of the 1920's was the following:

Problems about urns. Classical problems: problem of de Moivre die, meeting problem, sharing problem. Repeated trials, J. Bernoulli's theorem. Poisson approximation formula. Probability of causes. Bayes theorem.<sup>19</sup>

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<sup>18</sup>Dějiny statistiky, Technika statistického šetření, Popisování souborů z hlediska kvalitativních a kvantitativních znaků. Míry rozptylu. Teorie indexních čísel. Tabulky úmrtnosti a jejich konstrukce. Míry intenzivní a časové; jejich výpočet z daného materiálu. Základy počtu vyrovnávacího. Metody statistického bádání o příčinných spojeních. Stabilita statistických čísel. Stochastická závislost mezi kvalitativními znaky a mezi kvantitativními znaky. Rozbor časových řad.

<sup>19</sup>Urnenaufgaben. Klassische Probleme: Würfelproblem v. Moivre. Rencontreprobl. Teilungsproblem. Wiederholte Versuche, Theorem v. Jakob Bernoulli. Näherungsformel v. Poisson. Wahrscheinlichk. von Ursachen. Theorem v. Thomas Bayes.

During the war, lectures on probability theory were given by Fuhrich who taught actuarial mathematics as well as statistics. The lectures were common for the Technical University students and the German University students.

The last technical university in the area of Czechoslovakia in which the actuarial technical course was offered, in 1908, was the German Technical University in Brno.<sup>20</sup> There had been no special lectures on probability theory or mathematical statistics given at that university until the end of 19th century. In 1906 Friedrich Benze (1873–1940), an assistant of mathematics, was charged with giving such lectures as a paid docent. Benze, who did not publish any work was teaching probability and statistics until 1939. The syllabus of his lectures remained the same all that time, in the academic year 1914/15 it was the following:

Probability Theory I, winter semester 2/0:

Mixing, divisions and composition of finite sets. Mean determination. Quotas, argument, intersection and dispersion (?). Distributions in Bernoulli scheme, Poisson scheme, Compounded Poisson scheme.<sup>21</sup>

Probability Theory II (and applications), summer semester 3/0:

The urn model and its numerical elements: Bernoulli scheme, Poisson and generalized Poisson urn models. The urn model. Bayes theorem. Mathematical risk theory. Theory of fair prices. Comparison between the theoretical urn model and the empirical distribution : theory of errors, mathematical theory of mass phenomena.<sup>22</sup>

Mathematical Statistics, winter semester and summer semester 2/0:

Theory of collective measure. Formal theory of statistical series. Formal theory of population. Biometric functions. Mixing of statistical distributions. Immigration and emigration. Statistical theory of dispersion of sets. Building of statistical tables: Interpolation and compensation of mortality and disability tables. Local and analytical compensation.<sup>23</sup>

Actuarial mathematics was taught by Ernst Fanta (1878–1939), an actuarial mathematician from Vienna, in 1906–1920. He had worked in the seminar of Georg Bohlmann (1869–1928) in Göttingen in 1901–1902. During the 1920's actuarial mathematics was taught by Ferdinand Schnitzler (1857–1933), and then, until the war by Oskar Kubelka (1889–?). Despite great efforts of the teaching staff, the professorship of actuary mathematics was never established in Brno.

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<sup>20</sup>Details about mathematics teaching at the German Technical University in Brno and at universities in Prague are dealt with in [71]. Such course was never introduced at the Czech Technical University in Brno.

<sup>21</sup>Unordnungen, Zerlegungen und Zusammensetzungen endlicher Mengen. Mittelbildung. Quotenmittel, Argument, Durchschnitt und Streuung. Verteilungsgesetze im Bernoullischen, im einfachen und zusammengesetzten Poissonschen Schema.

<sup>22</sup>Das Urnenschema und seine numerische Elemente: Bernoullische, einfache und verallgemeinerte Poissonsche Urnenschema. Das verbundene Urnenschema. Bayessche Satz. Theorie des mathematischen Risikos. Theorie der Wertgleichungen. Vergleich des theoretischen Urnenschemas mit beobachteten Verteilungen: Fehlertheorie, Mathematische Theorie der Massenerscheinungen.

<sup>23</sup>Kollektivmasslehre. Formale Theorie statistischer Reihen. Formale Bevölkerungstheorie. Biometrische Funktionen. Mischung von statistischen Verteilungen. Ein- und Auswanderung. Dispersionstheorie statistischer Mengen. Konstruktion statistischer Tafeln: Interpolation und Ausgleichung von Sterblichkeits- und Invaliditätstafeln. Lokale und analytische Ausgleichung.

## 2. TEXTBOOKS

The 1860's was the time of a fast development of both secondary school and university education in Czech language in the Czech lands. In 1869 the Prague Technical University was divided into the German and Czech Technical Universities, and a similar division happened within the Prague University in 1882. The first Czech mathematics textbooks were written for the secondary schools to replace the existing German books. A significant role was played by the Union of Czech Mathematicians and Physicists from the very beginning as they paid attention to publishing secondary schools as well as university textbooks. The first task was naturally to cover the area of basic mathematical fields which meant the basis for the education of secondary school teachers, then engineers. As probability theory was not among those subjects, we cannot find any Czech book devoted to this topic until 1921. The situation was similar in other fields of higher mathematics in which German textbooks continued to be used and the specialists studied also the literature in other foreign languages.

The first Czech textbook on probability theory was the book *Počet pravděpodobnosti (Probability Calculus)* [42] from 1921. This book of Láška's has only 128 pages, including appendices, which are divided into three chapters: a priori Probability, a posteriori Probability, and Geometric probabilities. As E. Schoenbaum stressed in quite a critical review [69] in *Časopis* "a characteristic feature of the book is the author's interest in philosophical and noetic side of the calculus<sup>24</sup> and beside this he uses symbolic algorithms in an unusual scope; an undisputable advantage of this method is, however, balanced by the loss of space necessary for deducing the rules for calculations with the symbols for which the author introduces new, sometimes perhaps too complicated signs." Láška mentions also the need for axiomatization of probability theory in the introduction:

*From the point of view of pure mathematics our considerations on probability theory should in fact begin in a similar way as Hilbert starts his geometry. "Let us think about a concept which we name mathematical probability. We do not know what its transient meaning is, nor we have to or need to know either. Indeed, it would not be good if we wanted to know. All that is necessary to know about the concept will be told by the axioms." Unfortunately the axiomatics of probability calculus has not been worked out in a way to be able to present the introduction to the theory.*

Láška mentions the axiomatization attempts of Borel, Broggi, Bohlmann<sup>25</sup> and gives a short exposition of von Mises' collectives theory of 1919, insisting on the second Mises' axiom as a requirement for a "perfect mixing" of the sequence. *The main obstacle of Mises' theory lies in the mathematical formulation of the perfect mixing, and this, as it appears, and as Mises's more than extensive works prove, is not an easy matter.* In another place, Láška writes that *Mises' mathematical theory is in fact an analysis of geometric probabilities. This is its theoretical value, but also a mistake in the sense of application.*

Láška's book was published in the edition of the Czech Technical Matrix (Organization) in Prague, not by the Union of Czech Mathematicians and Physicists. There was an agreement between these two companies that, due to a limited number of Czech readers, they would not

<sup>24</sup>The footnotes prove Láška's good knowledge of older as well as more recent literature which is devoted to the development and philosophical questions of probability theory.

<sup>25</sup>Broggi and Bohlmann will also appear in Rychlík's textbook - see Part 4.



compete in publishing books on the same topic. That was the reason why in 1925 Hostinský's request for publishing his lectures on probability theory given in Masaryk University of Brno at the beginning of 1920's was refused by the Union with an explanation that his book would in fact deal with the same matters as Láška's textbook, with the exception of geometric probability. Hostinský was invited to write a booklet on geometric probability, which he did, and in 1926 his book *Geometric Probabilities* [29] was published.<sup>26</sup>

Hostinský wrote in the introduction:

*Various objections have been raised against the notion of geometric probability which is so important for physics (e.g. in kinetic theory of gases). Today, however, they are mostly of historical importance as the last decades have critically explained, mainly due to Borel, the concept of probability and its relation to physical applications.*

*This book has two purposes. First it gives the basic theorems on geometric probabilities and deals with exercises which are interesting from purely geometric point of view; a special chapter is devoted to considerations on attempts which can approximately confirm theoretical formulas for probabilities.*

Hostinský points out to a much more extensive book by Czuber [11] which he quotes on several places. He emphasizes that he uses for solutions some special exercises of Poincaré's "method of arbitrary functions" which is still not sufficiently known among Czechoslovak mathematicians.<sup>27</sup> Hostinský expresses his thanks to his assistant Josef Kaucký for the help with correcting the text in the introduction to the book. Kaucký belonged to Bohuslav Hostinský's pupils who were interested in the questions of theoretical physics as well as probability theory.<sup>28</sup> In 1934 he published the book *Introduction to Probability Calculus and Statistical Theory* [37]. The book was published from the initiative of electrotechnology professor V. List who belonged to the most important figures of Czechoslovak technical education between the wars, with the support of the Czechoslovak Electrotechnology Union. It is a small booklet (78 pages) in which Kaucký included besides the presentation of the basic concepts of probability theory<sup>29</sup> and mathematical statistics, also applications in the theory of errors and statistical mechanics. As Kaucký writes in the introduction, *The small scope of the book naturally resulted in the fact that I limited myself to the most elementary considerations at some places. Thoughts of philosophical nature and historical comments were also omitted.*

Another book which is interesting from the point of view of probability theory teaching in the Czech lands is *Základy teorie statistické metody (The Basics of Statistical Method Theory)* [38] written by Stanislav Kohn (1888–1933), a private statistics docent at the Russian Faculty of Law

<sup>26</sup>Hostinský made an effort to publish his lectures in lithography, but the Union refused. See the Masaryk University Archives, Bohuslav Hostinský Fond.

<sup>27</sup>Poincaré's method of arbitrary functions is introduced on pp. 70–85 where the reader is also acquainted with Hostinský's results of his work on a new solution of Buffon's exercise about the needle [28] or his work [30]. See also [27].

<sup>28</sup>Josef Kaucký (1895–1982) habilitated for mathematics at the University of Brno in 1928, he spent several years teaching at a secondary school in Brno, and in 1938 he was appointed mathematics professor at the Technical University of Košice in Slovakia which was transferred to Bratislava during the war. From 1946 he was teaching at the Technical University in Brno, and later at the Military Academy. He is known for his book *Kombinatorické identity (Combinatorial Identities)* within the Czechoslovak mathematical community. He was involved in the areas of difference equations, projective and differential geometry, and he also wrote several books on probability theory.

<sup>29</sup>Also Kaucký presented the method of arbitrary functions in his book.

in Prague. Kohn's book is based on his lectures given in Tiflis, Paris and Prague and it is dedicated to the memory of Kohn's teacher A.A. Čuprov. The book is nearly 500 pages long and a short analysis of approximately 100 pages on probability theory can be found in Mačák's book [45] (pp. 110–113). Kohn's book includes a big number of historical and especially philosophical comments, which makes it quite unique in the Czech literature. The list of the used and recommended literature is also very extensive, it comprises of more than 30 pages.

Another book, *Probability Calculus* [33], was written by Bohuslav Hostinský (1884–1951) in 1950, based on lectures given in the 1930's. In the introduction Hostinský points out that the lectures read in the 1930's differed considerably from those given in the 1920's. His expositions treat also the parts of his own scientific research. The first part includes a chapter on geometric probability and the second part consists mainly of the theory of Markov chains. The application in physics is not present there, Hostinský intended to deal with this topic in another book. This one, however, was not finished as Hostinský died in 1951. Hostinský's book does not include axiomatic building of probability theory, the reader is acquainted with the classical definition and then the author builds probability theory on the basis of theorems on addition and multiplication of probabilities which he calls axioms.

The only textbook, where an axiomatic construction is presented, seems to have been the one written in 1938 by Karel Rychlík, a professor at the Czech Technical University in Prague whom we shall present now.

### 3. KAREL RYCHLÍK: BIOGRAPHY

Karel Rychlík was born on 16 August 1885. His life is described in great detail in the book of M. Hykšová [34] which is the author's Ph.D. thesis at Charles University in Prague. The information given in this part of the paper is mostly based on that book.

Rychlík started his secondary school education in 1896 by attending grammar school in Chrudim, continuing in his native town Benešov from 1897, and finishing with a graduation exam at the Academic Grammar School in Prague in 1904 where the family had moved in 1900.<sup>30</sup> Rychlík had already showed a great interest in mathematics as a secondary school student and his name could be often found among those solving successfully the exercises for students published in the *Časopis pro pěstování matematiky a fysiky*. His mathematics teachers at the prestigious Academic Grammar School included Jan Vojtěch (1879–1953), his later colleague at the Czech Technical University in Prague.

Rychlík commenced his university studies in the winter semester of 1904/05, he studied at the Faculty of Arts of the Czech University in Prague which was then called Charles-Ferdinand University. It was the period of improved mathematics teaching at the university as the positions which had been vacated by the deaths of two old and ill professors, of professors František Josef Studnička (1836–1903) and Eduard Weyr (1852–1903), were being occupied by Karel Petr (1868–1950) and Jan Sobotka (1862–1931). It was especially the algebraist Petr who mostly influenced Rychlík's scientific work. Rychlík became a member of professor Petr's seminar, and he also gave lectures on his seminar work at the meetings of the Union of Czech Mathematicians and Physicist whose member he had become after starting the university. He received a scholarship of Bernard Bolzano Foundation for his outstanding study results in 1906.

In the academic year 1907/08 Rychlík obtained a national scholarship and studied at the Faculté des Sciences in Paris. He attended, apart from others, lectures of Jacques Hadamard, Emile

<sup>30</sup>Two years before Bohuslav Hostinský had graduated at the same grammar school.

Picard or Gaston Darboux.<sup>31</sup> At the Collège de France he listened to lectures on number theory which were given by Georges Humbert. During his Paris stay Rychlík already worked on his thesis which he submitted at the Prague University in 1908. A part of the thesis was published in the *Časopis pro pěstování matematiky a fysiky* with title *On a Group of order 360 (O grupě řádu 360)*. After a successful *rigorosum examination* Rychlík was awarded the degree Doctor of Philosophy on 30 March 1909. Before that, he had passed also the *teacher examination* for teaching mathematics and physics at secondary schools and he did his teacher practice year at grammar school in Žitná ulice, Prague.

From the beginning of 1909 Rychlík was helping in the mathematical seminar of the Faculty of Arts as a non-paid assistant. He worked as a paid assistant from October 1909 till the end of June 1913 when he became an assistant at the department of mathematics with professor F. Velíšek (1877–1914) at the Technical University.<sup>32</sup>

In November 1910 Rychlík submitted at the university an application for granting *veniae docendi* from mathematics with a habilitation work *A Contribution to the theory of forms (Příspěvek k teorii forem)*. Probably due to his quite small publication activity it was only in March 1912 that his habilitation procedure was successfully finished and Rychlík was appointed a *private docent* (university lecturer) of mathematics. In the academic year 1912/13 he started reading lectures at the university. He stopped with regular lecturing only in 1925 when he was a professor at the Technical University. Rychlík continued giving irregular lectures at Charles University until World War II. Let us remind that in the years 1912–20 another private docent working at the university was Bohuslav Hostinský. During the World War I they alternated (together with Bohumil Bydžovský (1880–1969)) in giving introductory lectures on differential and integral calculus and analytical geometry in space. As a private docent Rychlík usually read lectures on algebra and number theory topics, which were the fields of his own mathematical research.

When Rychlík had come to the Technical University he applied for transferring his habilitation from the university and he became a private docent at the Technical University too. While professor Velíšek was immediately after the beginning of World War I called up and died in the war, Rychlík was never called up and during the war substituted all Velíšek's basic lectures including the theory of probability. It seems very likely that had it not been for the given reasons, Rychlík would have not found a way to probability theory.

The formation of Czechoslovakia brought significant changes also in university education. The time came to fulfilling a long lasting attempts to create the second Czech university when in 1919 Masaryk University in Brno was founded. The Czech Technical University in Prague was re-organized. These two events resulted in increasing the number of mathematics professor positions. Hostinský, as well as Rychlík were suggested to be appointed professors by the board of professors at Charles University in March 1919, however the appointment was never put into practice. Hostinský was appointed full professor of theoretical physics at Masaryk University in 1920, and Rychlík was appointed extraordinary professor at the Prague Technical University in the same year. He was appointed full professor there in 1924.

Rychlík worked at the Technical University until the closure of Czech universities by the Nazis in November 1939.<sup>33</sup> During the whole period he gave lectures especially on differential and

<sup>31</sup>A year later, it was Bohuslav Hostinský who studied in Paris.

<sup>32</sup>Karel Rychlík succeeded his brother Vilém in that position. Vilém Rychlík (1887–1913) was a talented mathematician who unfortunately died very young, at the age of twenty six.

<sup>33</sup>In the academic year 1934/35 he was the Dean of the Faculty of Mechanical and Electrical Engineering.

integral calculus, and those parts of high mathematics necessary for future engineers. His scientific interests would certainly suit a university professor post, however, at Charles University the number of teachers grew very slowly around 1920, and algebra and number theory, the two main fields of Rychlík's interest, were covered by Karel Petr until WW II.

After the closure of Czech universities, like all other professors Rychlík was sent to the so-called expectant leave which meant his existing salary. The possibilities of scientific work were limited during the war mainly for an extremely restricted approach to literature, new as well as old – locked in university libraries. Rychlík prepared for publishing the second edition of his textbook on elementary number theory and in 1944 started translating Glivenko's textbook on probability theory [25].

Rychlík did not return to the Technical University after the war. He had been accused and in May 1946 found guilty of *breaching the loyalty to the Czechoslovak Republic and transgressing against the national honour* by the cleansing committee of the Provincial National Committee in Prague. Based on this conviction Rychlík was punished by being made permanently retired with a pension reduced by 30%. Rychlík appealed to the cleansing committee at the Ministry of Education which, however, rejected his appeal at the end of 1947. The reasons why Rychlík was made retired are given by Hykšová in [34] – e.g. according to the opinion of Vladimír Kořínek (1899–1981), Rychlík's assistant in 1927–1931 and later professor at Charles University, *approximately from the beginning of the civil war in Spain, Karel Rychlík began to show fascist opinions and his unconcealed admiration for Fascist regimes. This even increased during the so-called second republic.*<sup>34</sup> Other witnesses in the trial with Rychlík pointed out that Rychlík approved of many measures of German organisation and administration, and that he manifested his belief that the Germans were going to win the war. According to the committee, Rychlík must have been aware that such words were unsuitable to be expressed by a university professor in public in the time of persecution of the Czech nation.

Rychlík spent the rest of his life, especially the period 1953–58 when his retirement pension was drastically reduced, in poverty. The only activities he was allowed were connected with the work in the history of mathematics, especially treating the work of Bernard Bolzano. Apart from that Rychlík translated four books of Soviet mathematicians into Czech, the first one being Glivenko's textbook on probability theory. He died on 28 May 1968.

Hykšová divided the scientific work of Karel Rychlík into five groups: algebra and number theory (22 works), mathematical analysis (7), textbooks, popularization works, translations (16), works on Bernard Bolzano (14), and other works on history of mathematics (29). The most significant part is algebra and number theory in which Rychlík dealt with topical questions of building modern algebra. Most of his articles remained unnoticed in the world because they were published in national journals. The attention was drawn mainly to the paper *Zur Bewertungstheorie der algebraischen Körper* from 1923 which was published in *Journal für die reine und angewandte Mathematik* and was cited many times.

As far as mathematical analysis is concerned Rychlík published several works bordering algebra and number theory. He was one of the first mathematicians to deal with  $p$ -adic analysis.

Rychlík is an author of three textbooks the most important of which is *An Introduction to Elementary Number Theory* [64], 1931 which replaced the more than fifty-year old textbook of

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<sup>34</sup>This was the period from 1 October 1938 until 15 March 1939. Kořínek also states that he later reduced his contacts with Rychlík, and so he cannot say anything about his opinions during the war. It is probably an attempt to save Rychlík from a harsher punishment.



Studnička. It was published in 1950 in a second revised and enlarged edition. Rychlík's textbook on probability theory [67] is treated in Part 3 of this paper. The third Rychlík's textbook [68] from 1957 deals with the methods of numerical solution of algebraic equations with real coefficients and it is based on his university lectures.

From the point of view of Czech mathematics, Rychlík was famous, apart from his textbook on elementary number theory, for his interest in history of mathematics. Except for several popularizing articles about Czech as well as foreign mathematicians,<sup>35</sup> Rychlík made a name with his scientific work on part of Bolzano's inheritance. His interest in this area can be already seen in the 1920's when he published a paper dealing with Bolzano's continuous function which has a derivative at no point. Rychlík was a member of the Bolzano Committee under the Royal Bohemian Society of Sciences from 1924 and he took part in publishing Bolzano's works *Functionenlehre* (1930) and *Zahlentheorie* (1931). He continued to be a member of the Committee after the war, until 1952 when it was suppressed. It was, however, restored in 1958 under the Czechoslovak Academy of Sciences, though it existed only briefly, until 1961. During this period no large Bolzano's works were published and Rychlík concentrated on writing comments on Bolzano's works connected to the theory of real numbers and logic.

The work of Rychlík's on Bolzano's inheritance was famous internationally, he read a lecture *La théorie des fonctions de Bolzano* at the international mathematical congress in Bologna in 1928.<sup>36</sup> After the war Rychlík corresponded with some significant Bolzano researchers.

Karel Rychlík was an extraordinary member of the Royal Bohemian Society of Sciences from 1922 and an extraordinary member of the Czech Academy of Sciences and Arts from 1924.

#### 4. RYCHLÍK'S TEXTBOOK

Rychlík's textbook [67] is a booklet containing 144 pages. It is not printed but was produced using a reprographical method by zincography, an ancestor of the offset printing where a 'photography' of a text was obtained using the static electric properties of paper covered with zinc oxydium. The booklet presents therefore itself as a typed manuscript. All the special signs are hand-written, in particular all the mathematical signs and formulae: this tedious work had been realized by Otokar Pankraz, who is acknowledged for that in the foreword. Pankraz was Rychlík's assistant at the Technical University and we shall return to him at length in section 5. The text is divided in chapters (with a main title), and in numbered sections and sub-sections from 1 to 47, immediately followed by complementary sections about set theory numbered from 101 to 104. This strange gap in numbering may be an indication of the author's intention to complete and improve the text which is in the present state only a kind of first draft of a future book. With this respect, the text has the exact aspect of lecture notes written by a teacher for students studying probability theory. And the title itself where the word *Úvod* (Introduction) is highlighted contributes to the fact that the book is intended for beginners or at last for undergraduate students, who possess only the general mathematical bases, mostly in Analysis and Calculus, such as general properties of real functions and basic (Riemann) integration. The book contains a rather large bibliography, and also an index of the terminology, two properties not so common at the time (certainly not in textbooks devoted to students), which at least proves the author's remarkable modern pedagogical concern.

<sup>35</sup>Let us recall at least his works concerning Abel's and Cauchy's stays in the Czech lands.

<sup>36</sup>Karel Rychlík took part only in two international mathematical congresses, apart from the one in Bologna, he was in Strasburg, 1920.



Although the book was published as a textbook for technical university students, and both Hykšová [34] and Mačák [45] state so, there might be arguments that indicate that the text was not intended primarily for technical universities. First, the exposition is not usual for technical university students, in the sense that it is too abstract, e.g. as far as we know the set theoretical conception of mathematics did not appear at technical universities in Czechoslovakia before WWII at all. Rychlík's option for this kind of presentation can be explained either by his highly innovative approach, or by the fact that he had a different audience in mind. The latter reason can be explained by another argument: Rychlík wanted to use the text for his lectures at Charles University, however, he used the opportunity to publish it with the support of the Central Publishing Committee of the Czech Technical University.

The author states in the Foreword of the book that his aim is to compare axiomatic theory with other possible presentations of probability, nevertheless, he mainly concentrates on the axiomatic presentation. Let us therefore first compare Rychlík's title to the two other textbooks where Kolmogorov's axiomatic definition of probability was exposed before, namely Kolmogorov's own original text [40] and Cramer's textbook [9]. In Kolmogorov's title, the word *Grundbegriffe* (foundations) is a clear allusion to the word *Grundlagen* in Hilbert's famous treatise on the axiomatic foundations of geometry, and as such should have been understood by the mathematicians of the time as not devoted to students' use. This was also of course obvious from Kolmogorov's choice of Springer's collection *Ergebnisse der Mathematik und ihrer Grenzgebiete* for publishing his memoir, a collection that collected the most modern aspects of mathematics in progress and was clearly intended for specialists only. As for Cramer's book, its title is clearly much more technical as it begins by the words *random variables* and therefore also could certainly not reasonably be formulated for undergraduate students.

Therefore, as Hykšová [34] had already mentioned, Rychlík's textbook seems to be a quite rare document about an attempt to teach an axiomatic version of probability theory to undergraduate technical university students in the 1930's. Rychlík chose a rather abrupt axiomatic approach of probability, using set theory, which could justify the argument that the textbook was intended for his Prague University students as technical university students were not familiar with even basic aspects of modern set theory. The author added a special part at the end of the booklet (from page 114) called *Množiny, Funkce, Tělesa množinové* which is to say Sets, functions, set fields (the name used for Boolean set algebras). In the first paragraph, he presents the basic properties of sets, beginning by a short presentation of the concept following Cantor's ideas. And indeed the section begins with a quotation of Cantor's assertions about what a set is from his paper [6]. The same quotation appears in the subsequent Pankraz's paper [57] that we shall describe in the next section.

He also refers to the new Čech's book on set theory [7] advising the interested reader to consult it and the bibliography it contains. Čech was then professor of mathematics in Brno (where he had replaced Matyáš Lerch, a world famous specialist in number theory, after his death in 1922), and a prominent specialist in topology. As Frolík states in the Foreword to the third edition of Čech's book from 1974 (this edition is enlarged by the chapters which were found after Čech's death in the 1960's), this book played a very important role in introducing modern set theory in Czechoslovakia. Before there had not been even well-established Czech terminology concerning set theory. The book is amazingly dense, even for today's reader, and contains almost only the technical aspects of the subject, here again in a style which may certainly remind of Bourbaki's philosophy of mathematics. In its first edition, it contained four chapters, the last one devoted

to abstract measure theory, where Čech introduced algebras and  $\sigma$ -algebras and measures built on them. In particular, he builds Lebesgue measure over  $\mathbb{R}^n$ . This part on measure theory disappeared in the third edition of Čech's book.

Rychlík insists on different operators on sets he would use continuously: inclusion, union and intersection (denoted as the sum  $A + B$  and the product  $AB$  of sets  $A$  and  $B$  as was usual). The paragraph about functions (pp.118-121) is quite amazing for a textbook written for students non specialized in mathematics. Rychlík gives the set-theoretical definition of function from set  $\mathcal{U}$  to set  $\mathcal{V}$  as a subset of the graph  $\mathcal{U} \times \mathcal{V}$ . Certainly, he would not have expected his students to read or at least to assimilate these notions. The fact that this part is situated in the last pages of the book may be a testimony of this. Nevertheless, he may have been intellectually satisfied to show the possibility of an entirely axiomatic construction based on set theory, a point of view selected in the same years by the founders of the Bourbaki group for the redaction of their *Eléments de Mathématiques*. This also allows him to make an allusion on the notion of cardinality, defining countable and uncountable sets (pp.121-122), a difference which would play a role in his exposition. This section is followed by the definition of complement sets and a presentation of de Morgan's algebraic manipulation of sets, a paragraph on combinatorics, and one about the *characteristic* (indicator) function of a set. Then comes the final section numbered 104 called *Tělesa* (fields). Rychlík defines Boolean algebras on  $E$ , as a set  $\mathcal{A}$  of subsets of  $E$  which is stable by sum and difference, and gives elementary examples. As he is quite exclusively concerned by probability theory in countable sets (and mostly in finite sets), he can reasonably limit himself to this situation. It is however not completely true as he also wants to give some considerations on real random variables. He therefore adds a last paragraph (104,4) where he defines a Boolean algebra on the interval  $[A, B]$  as containing the empty set and all the sets which can be written as a disjoint sum  $I_1 + I_2 + \dots + I_n + \{p_1\} + \dots + \{p_k\}$  where the  $I_j$  are intervals (of any kind) included in  $[A, B]$ . He gives a correct sketch of the proof that this is indeed a Boolean algebra, which he denotes by  $\mathcal{A}(A \dots B)$ . It may be seem strange at first glance that Rychlík would not consider the usual situation (considered by Kolmogorov in his section 3) of the Boolean algebra  $\mathcal{B}_0$  of finite union of disjoint intervals of the type  $[a, b]$ . However,  $\mathcal{B}_0$  does not contain single sets  $\{p\}$  for which Rychlík would like to estimate probability when he considers real-valued random variables. Of course, Kolmogorov had no problem with this, as he immediately extends  $\mathcal{B}_0$  to the generated  $\sigma$ -algebra.

Now let us go back to the beginning of the book, which opens by a historical introduction. More precisely, this part is mostly concerned with philosophical considerations about the so-called classical definition of probability (the Laplacian definition as the number of favorable cases divided by the total number of cases) and the problems it generates, as well as the attempts by Cournot, Ellis and Venn to define only an *a posteriori* probability. Rychlík also mentions von Mises' construction of admissible sequences (*Kollektiv* - collectives: the word seems, however, not to be written down in the book) having correct limit relative frequencies, and he refers to Copeland's paper [8], as well as to Kamke's book [36]. Hykšová [34] had already observed that Kamke's book seems to have had a great influence on Rychlík, which he had reviewed for the *Časopis pro pěstování matematiky a fyziky* and which had given him the occasion of his only research papers approaching probability theory. Moreover, after the appearance of Mises' book [46] in 1931, Rychlík read a lecture on probability theory based on Mises' considerations at Charles University in Prague.

At the end of the introduction to his booklet, Rychlík writes that *there are various ways of determining probability. All these ways are attempted to be included also in the axiomatic method by means of laying down certain theorems which we do not prove - axioms - and other theorems are deduced as their logical consequence. Using this method is common in some parts of mathematics; the first attempts for axiomatization of probability theory were given by Bohlmann and Broggi.* The names of Bohlmann and Broggi belong to the long tradition which in Germany took over Hilbert's sixth problem of axiomatization of probability and appear in particular in the text presented by Bohlmann in 1908 at the International Congress in Rome. This story is deeply studied in [70]. Finally, Rychlík concludes by asserting that he *will use Kolmogorov's axiomatization. Its usefulness is in expressing the results of the theory of probability by means of mathematical results: a random event is defined as a set of elementary events and the probability as a (real) set function.* One can observe that, basically, the justification given by Rychlík for the use of Kolmogorov's axioms is more or less the same as the one quoted by Cramer in [9]: it is a pure question of mathematical convention. Or, to say it more dramatically: *there is no justification at all.* As Cramer writes down: *the question of the [convergence of relative frequencies to the probability] will not at all enter into the mathematical theory.* Though the whole text is written at non very high level, it seems that Rychlík wanted to convince his reader that probability theory is a part of mathematics. It is usual, when teaching at this level, to have rather general considerations about randomness and random experiences, and above all to use such considerations to generate intuition in the reader's mind (and this is indeed the specificity of probability calculus inside mathematics). On the contrary, Rychlík tries to get rid of such aspects any time it is possible as was of course the case with Kolmogorov's own book. However, as already mentioned, the Soviet mathematician had another kind of audience in view. A remarkable example of the fact is situated in the chapter devoted to independence. To give an example of three events  $A, B, C$  such that  $A, B$ ,  $A, C$  and  $B, C$  are independent, but  $A, B, C$  are not, Rychlík repeats exactly Kolmogorov's redaction (page 10 of the *Grundbegriffe*, note 3) on a completely formal mode, without expressing a *concrete* situation – e.g. with dice – though it generally makes the result obvious to students. Poincaré for example in [49] does not give other formulations than through card games, dice or urns models. Rychlík himself quotes Bohlmann's example from [4], which is described with help of numbered black and white balls in an urn.

The proper lecture notes open by 5 pages introducing the vocabulary of probability. This is an occasion for Rychlík to furbish the (today!) usual dictionary between set theory and probability folklore (set/event, disjoint sets/incompatible events and so on) which take place on pp. 8-9 and which he had directly copied from Kolmogorov's [40] section 3 *Terminologische Vorbemerkungen*.

The next part is called *Klasická definice pravděpodobnosti* (Classical definition of probability): Rychlík recalls Laplace definition and illustrates it by considerations on *Heads and Tails* and on dice. He immediately mentions (page 13, paragraph 6) that the definition is unsatisfactory. To illustrate this, he recalls the classical d'Alembert's 'mistake' where he obtained the value  $2/3$  for the probability of getting at least one *head* in two throws of a coin. D'Alembert claimed that it was sufficient to consider the set of events H, TH, TT as one may stop to play after having obtained one *head*. In fact, Laplace in [41] had already commented on d'Alembert's error and mentioned the necessary hypothesis that the elementary cases had to be supposed equally probable (and henceforth the definition of probability becomes circular. . . ). It can be also pointed out that placing an example of wrong probability calculations immediately after Laplace's classical definition

of probability was usual – see e.g. Láška [42], pp.7–8 and Hostinský [33], p.8, mention the example with throwing two dice, Kamke [36], p.15 gives d’Alembert’s error, and Czuber [14], p.738 states both of them. Nevertheless, as [45] suggests, Rychlík may have considered d’Alembert’s mistake so convincing that he placed it before introducing his axiomatic presentation.

Rychlík introduces his axiomatic presentation for a probability on a field  $\mathcal{A}$  with the axioms that are in the same order as in the first section *Axiome* of Kolmogorov’s booklet [40]. Like Kolmogorov, he gives as the first example the construction of probability on a finite set  $E$  by attributing a nonnegative number  $p_i$  to any element of the set  $E$  such that the  $p_i$  add to 1.

The following chapter, called *Zobrazení a ekvivalence pokusů a rozložení pravděpodobnosti* (Experiments mappings and equivalences and probability distributions) is an exposition of the transfer theorem. The presence of this rather theoretical chapter at this early stage seems mostly to offer a solution to the paradox mentioned above in the so-called classical definition of probability. Having mentioned the paradox, on page 13, Rychlík promised that in section 12,1<sup>37</sup>, he would provide material to solve it. The aim of the chapter is to emphasize the fact that for two random experiments to be stochastically equivalent, it is not sufficient that the probability spaces containing the elementary events be equivalent which is to say that there is a one-to-one transformation between them. In the paradox case, the probability spaces are the same. One has also to look at the behaviour of the probabilities on the algebras on each of the sets. Therefore, one defines the stochastic equivalence in the following way: the probability spaces  $(E, \mathcal{A}, P)$  and  $(E', \mathcal{A}', P')$  are equivalent if there is an application  $\varphi : E \rightarrow E'$  such that  $P'(A') = P(A)$  if  $A' = \varphi(A)$ . Though Rychlík’s presentation is a bit intricate, it is worth noticing that he tries to attract his students’ attention to the basic modern idea of probability calculus: one has to get rid of the probability space and to work with distributions. It is remarkable that this idea comes so soon in his textbook, before exposing the classical results and tools of the theory or presenting any application. We however do not know how the students of the Technical university may have received the fact.

On the contrary, the two chapters to follow (pages 27 and 37, respectively) are “expected”: the first one treats the (elementary) conditional probability, and the second one that of independence. The conditional probability of an event  $A$  given an event  $F$  such that  $P(F) > 0$  is defined as  $\frac{P(AF)}{P(F)}$ , and, apart from the fact that Rychlík observes that it satisfies the axiomatic definition of probability, the chapter contains the usual facts: computations in the case of uniform probability, Bayes formula and the classical illustration with urns models.

An interesting fact may be observed on page 33, where Rychlík presents Bayes’ formula. He mentions the usual term *cause of B* for an event  $A$  for which  $P(B/A)$  is given, but also adds that *for Fréchet, this event should be preferably called the hypothesis for the event B*. This sentence seems to be connected with Fréchet and Hallbwachs’s book [23], quoted in the bibliography. The title of Chapter III of this book is *Probability of hypotheses (or causes)*. Fréchet, keeping a statistical point of view on mind, sees the core of Bayesian method as the fixation of a priori probabilities. For instance, in the subsection *Precautions to take when using Bayes’ formula*, Fréchet insists on problems appearing *when the probability  $\pi$  of different hypotheses before the event are insufficiently known* (our emphasis). And he subsequently develops Poincaré’s example

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<sup>37</sup>On page 13, the marked reference is section 21,1, which does not exist. There has clearly been an inversion in the figures.



of determination of the status of a card player, cheat or fair player, following the type of card he presents.

The long chapter on independence (pp.37-56) contains rather tedious considerations on the independence of sets. Rychlík then applies them to introduce the binomial distribution and to check its general properties.

The next chapter is devoted to the notion of expectation of a random variable. Though he could have defined a random variable with more generality, Rychlík chose for the moment to restrict himself to variables taking only a finite number of real values, so that the definition of expectation is immediate, but only through the distribution, as Rychlík is not able to define it as an integral. He therefore cannot avoid the usual ambiguity of the axiomatic presentation for elementary probability. For example, to prove that the expectation is linear (more precisely, that the expectation of the sum of two random variables is the sum of expectations) costs him the usual tedious proof - deported later in 33 pages 77-79 in the chapter about the law of large numbers where he cannot avoid it. However, it is worth noticing that he makes use of the notions of equivalence and transformation introduced before to explain that all the important facts about a (finite valued) random variable may be explained over a finite space (see in particular the paragraph at the top of page 59).

But it is the next chapter, p.63, called *Spojité rozložení na přímce a rozložení spočetné* (Continuous probability on the real line and countable probabilities) when Rychlík must try to do something with his axioms, though he lacks the structure of  $\sigma$ -fields. It must be mentioned that he mostly gets out of the trap with elegance. As Rychlík does want to present continuous probabilities also in an axiomatic way, he makes use of the already mentioned Boolean algebra  $\mathcal{A}(A \dots B)$  on  $[A, B]$  composed by sets of the form  $I_1 + I_2 + \dots + I_n + \{p_1\} + \dots + \{p_k\}$ . If one considers a non-negative, non-decreasing function  $F$  such that  $F(A) = 0$  and  $F(B) = 1$  (Rychlík uses the same, most confusing, notation  $P$  for the function  $F$  and the probability built on it), a probability on  $[A, B]$  equipped with the mentioned algebra is thus defined by the following properties

$$P(\{p\}) = 0, P(I) = F(b) - F(a) \text{ if } I = (a, b).$$

Next (p.64), he asserts that *very often, the probability on the field  $\mathcal{A}(A \dots B)$  is defined through*

$$F(\alpha) = \int_A^\alpha p(t)dt$$

where  $p$  is a non-negative, (Riemann) integrable function on every finite interval in  $[A, B]$  and such that  $\int_A^B p(u)du = 1$ . As he is not able to properly define a real valued random variable, Rychlík defines only the *mathematical expectation of the function  $f(\alpha)$*  by the formula

$$E(f(\alpha)) = \int_A^B f(\alpha)p(\alpha)d\alpha$$

*if the integral exists.* He then introduces Gaussian and uniform distributions through their densities. The following paragraph (number 29) is devoted to the special case of a countable set  $E = \{\xi_1, \xi_2, \dots\}$ , where the probability is as usual taken as  $P(\xi_i) = p_i$  through a non-negative sequence  $(p_i)$  adding to 1. He takes care of clarifying that the Boolean algebra in this case is composed by all the subsets of  $E$ . This allows him to define a (real) random variable  $\alpha$  on  $E$ , simply as a function on  $E$  taking values in  $\mathbb{R}$ . The mathematical expectation  $E(\alpha)$  is then defined as the series  $x_1p_1 + \dots + x_np_n + \dots$  *if the series is absolutely convergent*. The justification given for this claim of absolute convergence is that it allows the result not to be ordering dependent.



In [40] (p.33), Kolmogorov defines the expectation as an integral in *the sense of Fréchet*: for  $x$  a random variable, if for any  $\lambda > 0$  the sum

$$S_\lambda = \sum_{-\infty}^{\infty} k\lambda P(k\lambda \leq x < (k+1)\lambda)$$

is *absolutely* convergent and  $S_\lambda$  converges to a limit when  $\lambda \rightarrow 0$ , this limit is the expectation of  $x$ . Therefore, integrability is of course equivalent to that of the modulus by construction. Kolmogorov does not mention the term order independence in the case of countable random values.

Maybe when he arrived at this point, Rychlík had the impression that he must tell his reader something about a more general situation, and to make him feel that this would require more technicalities which are not of the current level of the booklet. He adds an *Observation* (paragraph 29,3 p.72) where he formulates the continuity axiom for the probability  $P$  in the form of Kolmogorov's *Stetigkeitsaxiom* (p.13 of [40]): if  $(A_n)$  is a non-increasing sequence in  $\mathcal{A}$  such that  $\prod_{n=1}^{\infty} A_n = 0$  then  $\lim_{n \rightarrow +\infty} P(A_n) = 0$ . Rychlík does not mention the fact that in a Boolean

algebra it may occur that the set  $\prod_{n=1}^{\infty} A_n = 0$  were not in  $\mathcal{A}$ . He proves that if the algebra  $\mathcal{A}$

is finite, the axiom is straightforward. And adds that the continuity axiom is also *valid for the countable case*. In the case where the set of the elementary events is a Euclidean space the axiom [...] follows from the properties of Lebesgue measure for which he refers to the construction of Lebesgue measure in the aforementioned fourth chapter of Čech's book on set theory [7]. It is interesting to observe however that Čech treats the case of general abstract measures in the book, which satisfy the set continuity property (19.2.3 p.137) and not only the case of Lebesgue measure. Maybe Rychlík found too abstract for his students to imagine another measure than Lebesgue measure.

The last chapter is called *Posloupnostní model pro rozložení pravděpodobnosti* (Sequential model of probability spaces). It is an attempt to make the junction between the axiomatic point of view and sequential definition of probability in the style of von Mises's model of *Collectives*. The reader of Rychlík's text may feel uncomfortable with the rather abrupt transition with the rather linear exposition of axiomatic method followed by the author since the beginning, however, Rychlík points out in the preface that the book concerns also the relationship to older mathematical theories of probability. It is reasonable to think that Rychlík may have felt necessary to try to justify that the axiomatic definition did not prevent the common use of probability theory based on frequencies (such as in statistical classical models, such as mortality tables he gives as an example in section 46). As Shafer and Vovk had already observed [70], the German mathematical tradition (to which Czech scholars were still so close) made a clear distinction between theoretical probability (dealt with by mathematicians) and questions about its application discussed by experimental scientists and philosophers. Urban's book [73] (mentioned in Rychlík's bibliography) for example gives an interesting example presenting both aspects, but one after the other. Urban was an amazing universal mind, born in Brünn-Brno in Moravia in 1884 and whose thrilling life had espoused the shaken history of his country: his biography is narrated in [20]. See also Bru [5]. In 1923, he published the aforementioned book where he sums up his views on probability. The first part of the book (chapters I to III) deals with randomness and probability on a philosophical point. Urban proves there to have an extremely good knowledge of the literature

on probability and chance and in three cultural traditions (English, French and German) which seems to be an exception at the time. Certainly, his situation of member of a German community, in the Czech land and having been for 7 years in Philadelphia (USA) may at least partly explain this very large overview. The slightly smaller second part of the book (chapters IV and V) deals with the mathematical theory of probability. At first glance, it seems that there is no connection between the two sides of the book apart from the fact that the mathematical probability of an event is a real number attributed to this event, with rules inspired by our views on randomness. Urban writes in the introduction of chapter IV:

*The statements of the calculus of probability are abstract and are not propositions about real events. Their logic is internal, and nothing tells whether there exist objects corresponding to the conditions declared in these propositions. Neither with Bernoulli's statements nor with Poisson's theorem or any other more deep examination, it is possible to describe the reality. Every use of the calculus of probability in concrete situations must be preceded by an examination to check whether the studied processes and phenomena have the properties required by the theory.*

As already mentioned, Kolmogorov in the *Grundbegriffe* had evacuated the problem since the very beginning: he asserts that the question of the concrete interpretation of probability is not a mathematical question. To explain why Kolmogorov is so brief on the subject, it should also be noticed that he had already discussed this point before, and this time at length, in the introduction of his perhaps most important work in probability theory where he introduced continuous Markov processes [39]. However, Rychlík's purpose was to teach students, non specializing in mathematics, who moreover were supposed to be in contact with concrete financial application such as actuarial mathematics. Therefore, it was certainly necessary to say something. It is possible that the part about the sequential model is directly inherited from his former teaching of probability calculus following von Mises axiomatization that he prepared for Charles University in academic year 1931-32. In sections 37 to 41, he presents models of relative frequency of an event in a sequence, and shows that under reasonable hypotheses, the relative frequency limit satisfies the general properties required for probability. Rychlík develops the classical argument for foundation of probability theory on properties of relative frequencies. In Kolmogorov's booklet, the argument is briefly mentioned as *Empirische Deduktion der Axiome* p.4. The interesting part is section 42, called *Posloupnostní model pro rozložení pravděpodobnosti* (Sequential model of probability distribution). There he intends to join the axiomatic definition and the relative frequency model by defining a Boolean algebra as the collection of subsets for which the limit of relative frequencies exists. His main reference is Kamke's book ([36]) where Kamke formulates a (rather intricate) sequential model to define what he calls a *Wahrscheinlichkeitsfeld* as a sequence of sequences having good relative frequency properties. As said earlier, Hykšová [34] mentioned the importance of Kamke's book for Rychlík who seems to have been very convinced by the model. He wrote an enthusiastic review of the book in the *Časopis* ([65]). He mentioned there:

*while at the origin of the theory of relativity, the geometry of Euclidean and non-Euclidean spaces of three and more dimensions, which physicists needed, were already developed on a high level, mathematics are not in such a perfect state as far as probability theory is concerned. There are serious objections against the way how probability theory is presented nowadays. There is still much to be*

*done to have a perfect building of the theory and the consequences of the induced theorems are very often overestimated. However, as probability theory has gained a great importance, it is necessary to build it in the same clear and precise way as geometry is built. Nobody denies this need as far as presenting geometry is concerned.*

Rychlík produced soon afterwards himself a paper on well chosen binary sequences [66]: in the presently discussed chapter of Rychlík's booklet, the section 39 is apart from others devoted to the case of binary sequences of 0 and 1. The chapter ends by considerations about possible application of the sequential probability model, in particular with the already mentioned section (number 46) about mortality tables which may be an appealing example for students whose destiny should be to work in insurance companies.

## 5. RECEPTION AND DESTINY OF RYCHLÍK'S BOOKLET

It would certainly be extremely interesting to recover some impressions the students may have had by listening to Rychlík's lectures, as well as to know more precisely what he actually taught during these lectures. Unfortunately, it seems that we lack data for these facts. The main trace which is left is constituted by Pankraz's comments. Otomar Pankraz (1903-1976), already mentioned, was Rychlík's assistant at the Technical University.

Otomar Pankraz was born on 25 March 1903 in Nové Dvory near Písek.<sup>38</sup> His secondary school education was negatively influenced by a poor material situation of the family, which resulted in the fact that Pankraz could not finish a comprehensive secondary school, but graduated from a secondary technical school in Prague in 1923. This secondary school graduation exam did not allow him to attend a university as an ordinary student, he, therefore, followed the lectures at Charles University only as an extraordinary student in years 1923–1929. Only a supplementary secondary school graduation examination from 1929 enabled him to apply for doctoral examinations from mathematical analysis, algebra and theoretical physics. He obtained his doctoral degree in natural sciences in 1931. Although he had also fulfilled all the requirements needed for a future secondary school mathematics and physics teacher, he never took the final teacher examination.

In June 1929 Pankraz started working as an actuarial mathematician for preparing superannuation scheme in the General Pension Institute. However, it was only in 1931 that he passed actuarial mathematics examination and mathematical statistics examination with professor Emil Schoenbaum who worked as director of this institute in 1919–1939. Then Pankraz decided to follow an academic career with the possibility of scientific work and in May 1931 he became a mathematics assistant of Karel Rychlík at the Prague Czech University. He remained at this post until the closure of Czech universities and he was deprived of it officially only in March 1945.

In 1935 Pankraz habilitated at the university with the thesis *Zur Grundgleichung für den zeitlichen Zerfall der statistischen Kollektivs* [54, 52] in actuarial mathematics and mathematical statistics. According to Schoenbaum the work *shows the author's complete mastering of analytical tools for solving complicated integral-differential equations*. Pankraz's scientific activities were also favourably commented on by Karel Engliš, the governor of the National Czechoslovak Bank, and actuarial mathematics professors Löwy (Heidelberg), Moser (Bern) and Cantelli (Rome). During

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<sup>38</sup>The information about Pankraz's life were gathered from his personal files kept in the archives of Charles University, Prague Technical University and in the fund of the Ministry of Education in the National Archives in Prague.

his habilitation colloquium Pankraz was asked, apart from others, in great detail about Mises theory of probability.

From the summer semester 1936 Pankraz gave regular lectures as a private docent at the university, with the exception of the summer semester 1937 when he went to London for a research fellowship financed by *Rockefeller Foundations*. In the academic year 1938/39 he read lectures on probability theory instead of professor Kössler.

Based on the habilitation which took place in 1937 Pankraz was appointed a docent also at the Prague Technical University in 1938. He submitted two works on integral equations [53, 55] as his habilitation work. Positive opinions were given by professors Rychlík a Hruška who also praised other Pankraz's extensive publication activities. In 1938/39 Pankraz gave lectures on integral equations and their applications in engineering at the Technical University.

After the closure of Czech universities in November 1939 Pankraz lived in Prague and he received his assistant's salary until January 1945 although he did not perform any activity.<sup>39</sup> The payment of the salary was stopped after he had refused doing the work he was charged with by the Ministry of Education for more than a year.

Pankraz was arrested after the war, already on 19 May, and on 21 June 1946 he was found guilty of *propagating and supporting the Nazi movement especially by approving murdering university teachers after the execution of Heydrich and closing the universities and by praising the Nazism and the work of Czech traitors for Germany* by an extraordinary people's court. The court also managed to prove that the attorney-in-law Dr. Chlubna was arrested on the basis of his information. The court sentenced Pakraz to five years of heavy jail, the loss of property and 10 years of the loss of citizen's honour which he served in special working groups.

We know nothing about the later life of Otomar Pankraz except that he died in Prague on 12 December 1976 at the age of 73.

Pankraz, like Rychlík, seems to have been deeply interested in the evolution of probability theory during the 1930's. He regularly made reviews in the *Časopis*. In particular, it was him who reviewed von Mises' book: in his review [50], 1931, he not only describes the content of the book but also (briefly) presents the objections opposed to von Mises about his collectives theory and takes Mises' side. He writes:

*There are various objections against this theory. For example, it is said that it is not allowed to use the analytical concept of limit in probability theory. This objection is not valid because it is obvious from how Mises deduces the laws of large numbers that an analytical limit is absolutely satisfactory. However, the objections based on the principle of impossibility of gambling system and on the combination operation are more serious. I can see a solution to these difficulties in stating that Mises' requirements are principles which are closely connected to experience, not purely logic axioms (though derived from experience).<sup>40</sup>*

Pankraz had to face Hostinský's dissatisfaction about one of his reviews. Pankraz wrote in [51], 1932, the review of Hostinský's little treatise on Markov chains (his first really internationally celebrated publication) [31] and seems to have been deluded by the title *Méthodes générales* as he regrets that Hostinský had not given a general exposition of probability theory. Maybe the ambiguity of Hostinský's title explains also why it was the (young) Pankraz who had been given

<sup>39</sup>During the war he translated the book by E. Wagemann *Wo kommt das viele Geld her?* into Czech in 1943.

<sup>40</sup>For a discussion of early objections against Mises' theory, one may consult [74], pp.192-197.



the task of reviewing the text of an internationally known scientist as Hostinský. Despite the review is quite laudative, Hostinský seems to have been taken aback with it and subsequently wrote a rather dry answer, also published in the *Časopis* ([32]).

In 1938, Pankraz took over the task of reviewing Rychlík's booklet. Though reviews of that time were often longer than today as the reviewer included his own opinions, the length of Pankraz's review [56] is quite noticeable (two full pages and half: as a comparison the review of von Mises' book is only one page long), which at least proves a particular interest in the question (of course, self promotion might have been also involved!). Pankraz justifies the necessity of a book by the quick developments of the theory of probability, and mentions that Rychlík had chosen to exploit Kolmogorov's axiomatization. He adds that *contrary to Kolmogorov who (...) only briefly exposed the problems, professor Rychlík presents the axioms in detail*. This slightly exaggerated sentence can be understood as an indication that the levels of the two texts are very different. Then follows a description of Rychlík's textbook. Pankraz comments at length on the bases of the axiomatic foundation:

*The readers who are used to think with the concepts of classical probability theory must be warned that the transition to the new theory lies exactly in the definition of the random event. In the classical theory, this was a vague notion and its justification was based more on intuition than on mathematical bases. The core of the new theory is precisely to state the most important feature that it is necessary to consider a random event as a collection (i.e. a set) of events, not as an isolated event. There is also a conceptual difference between an element and a collection (set) of elements which is met nowadays in nearly every branch of pure and applied mathematics (and also logic) which makes the difference between "modern" and "classical" theories. Naturally, this difference has been known for a long time but is properly considered only in modern theories.*

After presenting the axiomatic definition of probability on a Boolean algebra, Pankraz adds that in his book Rychlík *shows very clearly and in details that it is indeed possible to build probability theory on the basis of these axioms and that classical probability theory is included in it. The transition to applications, mainly in statistics, is enabled by the so-called sequential model of probability distribution*. Once again, Pankraz describes the sequential model in details. The end of the review expresses a real enthusiasm:

*The set-theoretical considerations thus (1) provide us with an exact and simplified basis for building probability theory and (2) lead to an immediate application in statistics. At the same time, the results are more extensive than in the classical theory. The relevance of the new methods is indisputable and it now depends on didactical methods how it may be possible to transfer the way of thinking from the old form to the new one in the simplest way. It is only a question of schools and it is a good thing that the Czech Technical University in Prague is ready to support such new methods.*

*Rychlík's book, although it is modestly called an Introduction, presents author's original thoughts at numerous places and indicates directions in which the study of probability theory can be extended.*

One passage is rather remarkable for a review. Pankraz writes that *[T]he book is not printed, but realized with zincographic reproduction which enables the author to publish it later in a modified*



state which could reflect the new and latest results in probability theory. This method is much more suitable than publishing expensive printed books for the Czech environment where the sale of mathematical literature is rather small. What Pankraz intends with this sentence is not quite clear. Apart from the reasonable mention of the economic side of the question, the sentence may also suggest that the two men had discussed future possible improvements of the booklet. At least, the sentence seems to assert that the present state of the text is not completely satisfactory and that the work is in progress. Another hint of this is the fact that in winter term 1938-39 and summer term 1939, Pankraz had taught lectures on probability theory at Charles University and had written his paper [57] published in the Prague *Aktuárské vědy* (Journal of Actuaries). The paper may reflect the content of Pankraz's lectures, and is mostly devoted to the question of axiomatization of probability. For Pankraz, the two concurrent ways through which probability has been defined, i.e. Mises frequentist approach and Kolmogorov's axiomatic method generate unsolved problems. In Section 3 of his paper, he sums up these problems : 1) the internal contradiction of Mises theory which pretends to give a precise definition of probability though it uses concepts of subjective nature. 2) the incompleteness of Kolmogorov's axiomatization to which Kolmogorov is obliged to add a separate definition for conditional probability.

It is nevertheless clear in the paper that the accent is mainly put on Kolmogorov's axiomatic construction (quite remarkably, von Mises, though quoted in the text, is absent from the bibliography). Pankraz describes at length large bases of set theory in his section 5 (where, as already mentioned, he quotes Cantor's assertions on sets as Rychlík had done) and in section 6 connects these set-theoretical notions to the notion of random event. We had already observed in his review of Rychlík's book how he emphasized the fact that the set conception of the random event has become the basic notion of modern probability theory. The section 7 is entirely devoted to a brief description of Kolmogorov's axiomatics and to criticizing it in the light of his previous comparison of Mises and Kolmogorov definitions. For Pankraz, the introduction of conditional probability as a separate notion by Kolmogorov cannot be justified on a logical basis:

*In fact, the question is whether probability is defined as a function of one or two arguments. Kolmogorov tries to get along with the set function  $P(A)$  with one argument, defined through axioms. However, at the same time, he comes to the fact that this one-argument function is not enough for formulating Bayes formula, and therefore he introduces the set function*

$$P_A(B) = \frac{P(AB)}{P(A)}$$

*with two arguments  $A$  and  $B$ , which he proves also to satisfy his axioms. He must, however, suppose that the set  $A$  is fixed, and therefore this assumption allows him to see the two-argument function as a one-argument function which obviously satisfies the axioms.*

*Reichenbach has already pointed out that this method is not satisfactory. While Reichenbach's objection to the introduction of the new symbol  $P_A(B)$  was more formal than mathematical, we find the condition of  $A$  being constant in the definition of  $P_A(B)$  inadmissible from a mathematical point of view as in this case it would not be possible for  $A$  and  $B$  to be variable at the same time. ([57], [7,2]).*

Reading Pankraz's bibliography, it seems to have been quite interested by the discussions about the status of probability in quantum mechanics, which flourished during the first third of 20th

century. They have been the study of numerous papers: see for example [74], Chapter 4. A famous international conference organized by the philosophers from Berlin and Vienna Circles about the implications of the new physical theories was held in Prague (as a barycenter between the two towns!) in 1929. Here were present Carnap, Reichenbach, von Mises and many other of the names mentioned in the present paper. As Hykšová has already briefly suggested in [35], it is quite possible that Rychlík and Pankraz have also attended it. Pankraz had certainly at least read the texts of the conferences published in the founding issue of the journal *Erkenntnis*. In particular, he quotes Waismann's conference on the logical foundations of probability [75]. The two last sections of [57] are besides devoted to considerations about quantum mechanics.

The previous remark on Kolmogorov's axiomatization leads Pankraz to formulate his own system of axioms for a probability defined as a two-arguments set function, which he claims to be complete and non-contradictory. The probability is therefore defined as satisfying an additivity property on the second argument

$$P(A, B + C) = P(A, B) + P(A, C)$$

if  $B$  and  $C$  are disjoint sets (he mentions a version with  $\sigma$ -additivity on the next page), and the multiplication axiom as

$$P(A, BC) = P(A, B)P(AB, C)$$

when  $P(A, B) > 0$ . It can be seen that Pankraz formulates the same kind of axiomatization as Popper would find necessary to formulate only some twenty years later in the Appendix IV of [62]. It should be observed that in 1938, Popper [61] had proposed an alternative set of axioms where the primitive notion is the 'absolute' probability. In the Appendix II of [62], where the beginning of [61] is printed again, Popper asserts (with a little exaggeration) that he was the first to propose that the mathematical theory of probability should be elaborated as a *formal system*. Popper's paper is not mentioned in Pankraz's text though it was published in the journal *Mind* which Pankraz happened to know as it is quoted for Nagel's extensive review on Reichenbach's book in the bibliography of [57]. Maybe, the 1938 issue of the journal had not yet reached him at the time. However, he insists that *to speak about the probability of the event B without having mentioned the [reference] event A has no meaning*. Pankraz therefore joins the trend of those who consider that the basic notion is conditional probability, as estimating the probability of an event makes sense only with reference to another event: there is not such a thing as absolute probability. This leads some of the main representatives of this trend to refuse an axiomatic foundation, as de Finetti and his entirely subjective conception of probability. Much later, de Finetti exposed the following opinion in the introduction to Chapter 4 of his textbook on probability theory: *Every prevision, and, in particular, every evaluation of probability, is conditional; not only on the mentality or psychology of the individual involved, at the time in question, but also, and especially, on the state of information in which he finds himself at that moment* [21]. It should be extremely interesting to know what Rychlík or Pankraz, two seemingly open minded scientists who enthusiastically successively adopted von Mises and Kolmogorov's points of view on probability, may have thought of de Finetti's (being also an actuar like Pankraz was) views. Up to now, we have no trace to which extend Pankraz knew de Finetti's texts (such as his comments on Bayes theorem in [22], pp.20–23), the only place where he mentions his name is the review [60] of Nagel's book [47].

The next year 1940, Pankraz published a new paper, this time in the *Časopis* [58], where he came back to the fundamental notions exposed in his 1939 paper, to satisfy the claim of his students

looking for a smooth introduction to the new aspects of probability theory, as he himself justifies in the foreword ([58], p.D73). Quite interestingly, the name of Glivenko appears in the paper, through the book [24] published in Paris in 1938 by the Soviet mathematician about general algebraic structures where he mentioned some connection with the foundation of probability theory. In [24], p.26, Glivenko proposes to define the stochastic type of an event by considering that  $A \sim B$  if and only if  $P(A/B^c) = 0$  and  $P(B/A^c) = 0$ . On the *structure  $S$  whose elements are events, the probability defines a norm* (p.26). From Glivenko's book appears for instance for the first time in Pankraz's paper the expression Boolean algebra (p. D79). At that time, Glivenko had also written a textbook on probability theory [25], published in 1939 where he made use of Kolmogorov axiomatization. The book reached Rychlík somehow during WWII and was translated by him into Czech; it was published in Czechoslovakia only in 1950 ([26]). Basically, the paper presents also the construction of probability as a two-argument function. The text is however more strictly technical and Pankraz has left aside many of the philosophical questions he exposed in [57]. His aims seems to show how it is possible to rapidly construct a probability function with his axioms, which is practice to manipulate with. Nevertheless, up to this point Pankraz does not seem to have realized the problem in his system, where  $P(\emptyset, \emptyset) = 1$  and where  $X \subset Y$  may lead to probabilities greater than 1. In a subsequent note [59], Pankraz proposed a corrected version of his axioms. Not without some irony, he mentions that it was his reading of Reichenbach's book that lead him to this error. Naturally, one may reasonably comment that the obtained system of axioms, where it is now necessary to check the non-emptiness of the considered sets for the relations on probability to be valid is not as comfortable as was his former (incorrect) one. However, the paper [59] contains more than a simple technical correction, as it gives a separate status to the multiplication axiom. Pankraz's axioms are now the following ones

- (i)  $\mathcal{A}$  is an algebra of subsets of  $E$ .<sup>41</sup>
- (ii) For every couple  $(X, Y)$  of elements of  $\mathcal{A}$  such that  $X \neq \emptyset$ ,  $P(X, Y)$  is well defined and non-negative.
- (iii) For every couple  $(X, Y)$  of non-empty elements of  $\mathcal{A}$  such that  $Y \supset X$ ,  $P(X, Y) = 1$ .
- (iv) For any  $X, Y, Z$  in  $\mathcal{A}$  such that  $X \neq \emptyset$  and  $YZ = \emptyset$ , one has  $P(X, Y + Z) = P(X, Y) + P(X, Z)$ .

One may in particular observe that  $P(X, Y)$ , interpreted as *the probability of event  $Y$  given that event  $X$  occurs* is well defined even if  $P(X) = 0$  (unless  $X = \emptyset$  of course). This problem of the conditioning on 0 probability set had been for a long time a recurrent question. A celebrated example is the *great circle paradox* (sometimes called the *Borel-Kolmogorov paradox*) proposed by Bertrand among his famous list of paradoxes, and investigated by Borel in [3]: if a point is chosen *at random* on a sphere, the point is *uniformly* chosen on the equator but *not uniformly* on any meridian. The investigations around Bertrand's paradox have been studied in [70]. Borel has pointed that the status of the conditioning proposition, and that the source of the "paradox" was in the fact that at first glance, the conditions *the point is on the equator* or *the point is on a meridian* have probability 0. Kolmogorov, in [40] (p. 44), mentions Borel's paradox and claims that his own definition of conditional distribution via Radon-Nikodym theorem gave a satisfactory explanation for the paradox. Nevertheless, the impossibility of conditioning by a null probability event had never been considered as completely satisfactory, a classical observation being that

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<sup>41</sup>Pankraz's original notation for the algebra is  $\Omega$ !  $\mathcal{A}$  seems preferable for our comments, in order to avoid the nowadays compulsory understanding of  $\Omega$  as the set of elementary events.

it is hard not to be able to assert that  $P(X/X) = 1$ . Axiom (ii) of Pankraz gives indeed that  $P(X, X) = 1$  for any non empty set from  $\mathcal{A}$ .

Pankraz observes that his axioms (i) to (iv) are complete and non contradictory. However, up to this point, nothing allows to make a connection with a statistical interpretation. As it was the case in Rychlík's textbook as we had already commented on, this possibility of embedding the classical frequency interpretation of probability into any proposed axiomatic definition seems to have been a major concern of the two Czech mathematicians. Pankraz comments: *It is necessary to agree on the meaning of the expression 'statistical' interpretation. Generally one agrees that every number which can be reduced to frequencies is 'statistical'. Indeed, apart from a single case (quantum physics) this agreement never leads to doubts.* He therefore proposes his fifth axiom:

(v) For every couple  $(X, Y)$  of elements in  $\mathcal{A}$  such that  $X \neq \emptyset$ ,  $P(X, Y) = \frac{P(E, XY)}{P(E, X)}$ .

Pankraz should certainly preferably have written the axiom as  $P(X, Y).P(E, X) = P(E, XY)$ , to avoid a subsequent intricate justification of what happens if  $P(E, X) = 0$ . Then, asserts Pankraz,  $P(E, XY) = 0$  by axiom (iv) and so the fraction is not defined and this seems to mean that axiom (v) is not concerned by this case. Anyway, in the case where  $P(E, X) > 0$ , the quantity  $P(X, Y)$  is indeed equal to the quotient  $\frac{P(E, XY)}{P(E, X)}$ . Pankraz does not in fact explain why this quotient may be interpreted as a frequency: he had probably in mind a sequential model of the same kind as Rychlík's that we have presented in the previous section. From the comments he adds, it seems that his most concern was about the possibility or impossibility of building  $P(X, Y)$  from a common reference set, a question directly stemming from quantum physics:

*The question of validity of axiom (v) for quantum physics remains undecided. Is it possible to interpret every case of this physics by means of frequencies if we consider that there exist*

1. *the least measurable length of order  $10^{-13}$  cm*
2. *the least measurable time interval of digit place  $10^{-13}$  cm/c ,  $c$  = velocity of light in the vacuum, and*
3. *complementary measurable data following Heisenberg uncertainty principle?*

*Is it justified to add axiom V to axioms I – IV for the 'probabilistic' description of subatomic events taking place in spatio-temporal fields of order smaller than  $10^{-13}$  where, in general, I cannot speak about the possibility of 'counting' elements though these are collective events? As it is obvious, the question is whether the probability exposition of specific events necessarily means the exposition by means of frequencies, or whether its framework is wider.*

Pankraz concludes his paper by formulating the Bayes formula with his two-argument probability axiomatization.

## CONCLUSION

In March 1939 German troops entered Prague, and after few months of occupation, the Germans decided to close the universities. This moment marked also the end of the original teaching experience by Rychlík and Pankraz. In the first place, as was narrated before, the two had stopped their mathematical activity soon after the war. In the second place, Kolmogorov's axiomatization



gradually imposed itself as the most practice presentation of probability theory and therefore became more and more adopted. In the 1960's, almost every mathematician involved in probability adopted it. Exceptions may nevertheless be quoted. The 'old' probabilists, such as Paul Lévy, refused to change the presentation they were used to - a thing which seemingly did not prevent Lévy from brilliant discoveries on Brownian motion during the 1950's (see [44])! In Italy also, Kolmogorov's axiomatics was slow to appear, due to the obstinate opposition of de Finetti and his great personal influence in the Italian community. And of course, we may mention the strong opposition in France to the abstract measure theory by the tenors of the Bourbaki group. However, in Czechoslovakia after the installation of a Soviet Union inspired government, the situation was clear and Kolmogorov's axiomatization imposed itself as in the USSR. Rychlík even obtained some consideration from the new oriented politicians for having translated Glivenko's book from Russian before 1945. As already mentioned, the publication of his textbook on axiomatic probability theory in 1938 had been a surprising event without real consequences. But we do not know what they may have been in a quieter context. . .

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